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HYPERSONIC RAREFIED FLOW PAST AN INSULATED FLAT PLATE WITH SUCTION/INJECTION*

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Abstract—Von Kármán–Pohlhausen method has been used to study the rarefied hypersonic flow past an insulated flat plate with slip velocity and suction or injection on the surface. Analytical solutions are obtained for the leading edge and strong interaction regions. The governing equations are, then numerically integrated. Numerical results agree with the analytical results in the leading edge and strong interaction regions and provide solution in the intervening zone. Effects of slip and suction or injection on various flow quantities are discussed.

NOMENCLATURE

x, y,	distances measured along and perpendicular to the plate length;
u, v,	velocity components along x, y directions;
p, ρ, T,	static pressure, density and temperature respectively;
μ,	coefficient of viscosity;
C_p,	specific heat at constant pressure;
H,	$C_pT + \frac{1}{2}u^2$;
k,	thermal conductivity;
M,	Mach number;
λ,	mean free path;
δ,	boundary-layer thickness;
u_b,	slip velocity on the surface;
τ,	$\mu_w(\partial u/\partial y)_w$;
c_f,	$\tau/\frac{1}{2}\rho_\infty u_\infty^2$;
R,	gas constant;
c,	$\mu_w T_\infty/\mu_\infty T_w$, Chapman–Rubesin constant;
Pr,	$\mu C_p/k$, Prandtl number;
Re_x,	$\rho_\infty u_\infty x/\mu_\infty$;
ξ, η, L,	defined in (2.12);
$\bar{\chi}$,	$M_\infty^3(\sqrt{c})/\sqrt{Re_x}$;
θ,	$d\delta/dx$, local slope of the boundary-layer edge with x-axis.

Subscripts

∞,	conditions in the free stream;
w,	conditions at the wall;
e,	conditions at the edge of boundary layer.

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1. INTRODUCTION

IN HYPERSONIC flow over slender bodies, boundary layer becomes thicker and also the temperature on the body becomes higher with increasing Mach number. Suction may be necessary to reduce the interaction of boundary layer with the external flow while surface injection may be needed to alleviate the heat transfer to the body.

Shen [1] investigated the hypersonic flow on the wedge with no slip and temperature jump boundary conditions using von Kármán–Pohlhausen method with linear profile for the tangential velocity component. He found that the boundary-layer thickness varies as $x^{\frac{3}{4}}$ where x is the distance from the leading edge and the pressure ratio (p_w/p_∞) varies as $(x)^{-\frac{1}{2}}$. Bogacheva [2] modified Shen's [1] analysis to investigate the slightly rarefied hypersonic flow on a flat plate by taking slip velocity as the perturbation to the no-slip case. Later, Yasuhara [3] examined the strong interaction on a flat plate with suction or injection using a

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quadratic profile to represent the tangential velocity component in the boundary layer. He found that for similarity to be possible, suction or injection velocity must be proportional to $(x)^{-\frac{1}{4}}$. Li and Gross [4] also investigated the same problem using the method of similar solutions.

Here, we have investigated the hypersonic flow of a rarefied gas on an insulated flat plate with slip and suction or injection on the surface. We assume an inviscid flow in between the shock wave and the boundary layer. Tangent-wedge approximation provides the pressure distribution at the edge of the boundary layer. From similarity conditions, (v_w/u_∞) has been assumed proportional to $(p_w/p_\infty)^{\frac{1}{2}}$. Analytical solutions have been obtained for the leading edge region and for the strong-interaction region. Then, the governing equations have been integrated numerically. The numerical solution matches with the analytical solutions in the leading edge region and strong interaction region and provides solution for the intervening zone. A number of graphs have been drawn to illustrate the effects of slip and suction or injection. It is found that due to slip, pressure attains a finite value in the leading edge region. Suction increases the slip velocity while injection decreases it. In general, injection has larger effect on the various quantities than suction.

2. EQUATIONS OF MOTION AND THEIR SOLUTION

Equations governing a 2-dimensional steady flow on a flat plate are the following

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \tag{2.1}$$

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right) + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) \tag{2.2}$$

$$\rho u \frac{\partial H}{\partial x} + \rho v \frac{\partial H}{\partial y} = \frac{\partial}{\partial y}\left[\frac{\mu}{Pr}\frac{\partial H}{\partial y} + \left(1 - \frac{1}{Pr}\right)\mu \frac{\partial}{\partial y}(\tfrac{1}{2}u^2)\right] \tag{2.3}$$

$$p = R\rho T. \tag{2.4}$$

Boundary conditions are

(i) at $y = 0$, $v = v_w(x)$,

$$u = u_b = C_1 \lambda_w \left(\frac{\partial u}{\partial y}\right)_w \tag{2.5a}$$

where

$$C_1 \simeq \sqrt{(\pi/2)} \quad \text{and} \quad \lambda_w = 1{\cdot}256(\sqrt{\gamma})\frac{\mu_w}{a_w \rho_w}$$

(ii) at $y = \delta$, $u = u_e \simeq u_\infty$, $T = T_e$. (2.5b)

For $Pr = 1$ and slip velocity with no heat transfer at the wall, there exists the following integral of the energy equation (2.3):

$$H = H_e = H_\infty.$$

Neglecting terms of second order in λ_w, we get the adiabatic wall temperature as

$$\frac{T_w}{T_\infty} = 1 + \frac{\gamma - 1}{2}M_\infty^2 \simeq \frac{\gamma - 1}{2}M_\infty^2. \tag{2.6}$$

From equations (2.1) and (2.2), in the presence of slip velocity and suction/injection on the surface, we get the following von Kármán's integral:

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\delta \rho u(u_\infty - u)\mathrm{d}y = \frac{\mathrm{d}p}{\mathrm{d}x}\delta + \mu_w\left(\frac{\partial u}{\partial y}\right)_w + \rho_w v_w(u_\infty - u_b). \tag{2.7}$$

Last term in equation (2.7) shows that suction or injection interacts with the slip velocity at the wall.

Assuming

$$\frac{u}{u_\infty} = a + by \tag{2.8}$$

with boundary conditions (2,5a) and (2,5b), we get

$$\frac{u}{u_\infty} = \frac{C_1\lambda_w + y}{C_1\lambda_w + \delta}. \tag{2.9}$$

Substituting (2.9) in (2.7) and neglecting terms of second order of smallness (e.g. $C_1^2\lambda_w^2/\delta^2$,

$(\lambda_w/\delta)\cdot \mathrm{d}\delta/\mathrm{d}x$, etc.), we get,

$$\delta\frac{\mathrm{d}p}{\mathrm{d}x}+2p\frac{\mathrm{d}\delta}{\mathrm{d}x}=\frac{\mu_w u_\infty}{C_1\lambda_w+\delta}+\rho_w v_w u_\infty\left(1-\frac{C_1\lambda_w}{C_1\lambda_w+\delta}\right). \quad (2.10)$$

Substituting

$$\frac{p}{p_\infty}=\frac{\gamma(\gamma+1)}{2}M_\infty^2\theta^2=\frac{\gamma(\gamma+1)}{2}M_\infty^2\left(\frac{\partial\delta}{\mathrm{d}x}\right)^2 \quad (2.11)$$

from tangent-wedge approximation and non-dimensionalizing the variables in (2.10) as follows

$$\eta=\frac{\delta}{L},\quad \xi=\frac{x}{L}\quad \text{where}\quad L=\frac{\mu_w}{u_\infty\rho_\infty} \quad (2.12)$$

we get,

$$\eta\eta'\eta''+\eta'^3=\frac{1}{(\gamma+1)}\left[\frac{1}{(C_1\lambda_w/L)+\eta}+\frac{\rho_w v_w}{\rho_\infty u_\infty}\left(1-\frac{(C_1\lambda_w/L)}{(C_1\lambda_w/L)+\eta}\right)\right] \quad (2.13)$$

where prime denotes differentiation with respect to ξ.

Here,

$$\frac{C_1\lambda_w}{L}\left(\frac{\mathrm{d}\delta}{\mathrm{d}x}\right)^2=e \quad (2.14)$$

where

$$e=\frac{1{\cdot}256}{(\gamma+1)}C_1\sqrt{\left(\frac{2(\gamma-1)}{\gamma}\right)}.$$

For similar solutions,

$$\frac{v_w}{u_\infty}\propto\left(\frac{p_w}{p_\infty}\right)^{\frac{1}{2}}=AM_\infty\eta'\sqrt{\left(\frac{\gamma(\gamma+1)}{2}\right)} \quad (2.15)$$

where A is the constant of proportionality and is positive for injection and negative for suction. Using equations (2.14) and (2.15) in equation (2.13), we get,

$$e(\eta'^2+\eta\eta'')+(\eta\eta'^4+\eta^2\eta'^2\eta'')=\frac{b}{2}\eta'+\frac{ab}{2}\eta\eta'^4 \quad (2.16)$$

where

$$b=\frac{2}{\gamma+1},\quad a=\frac{[\gamma(\gamma+1)]^{\frac{3}{2}}}{\sqrt{2}}\cdot\frac{AM_\infty}{(\gamma-1)}.$$

Let

$$\eta=\xi^\sigma\sum_{n=0}^{\infty}a_n\,\xi^{nq}. \quad (2.17)$$

Substituting (2.17) in (2.16) and collecting terms of like powers of ξ, we find that for slip to be a dominant phenomenon near the leading edge,

$$\sigma=1\quad\text{and}\quad q=1.$$

Hence,

$$\eta=\xi(a_0+a_1\xi+a_2\xi^2+\ldots) \quad (2.18)$$

and

$$\frac{p_w}{p_\infty}=\frac{\gamma(\gamma+1)}{2}M_\infty^2(a_0^2+4a_0a_1\xi+4a_1^2\xi^2+\ldots) \quad (2.19)$$

where

$$a_0=\frac{b}{2e},\quad a_1=\frac{b^4}{28e^5}(ab-2),$$

$$a_2=\frac{a_0^4a_1}{9b}(9ab-2)-\frac{4a_1^2e}{3ab}.$$

For large ξ (strong interaction region), slip appears as a perturbation to the no-slip solution. This gives,

$$\sigma=\tfrac{3}{4}\quad\text{and}\quad q=-\tfrac{1}{4}.$$

Hence

$$\eta=\xi^{\frac{3}{4}}(a_0+a_1\xi^{-\frac{1}{4}}+a_2\xi^{-\frac{1}{2}}+\ldots) \quad (2.20)$$

and

$$\frac{p_w}{p_\infty}=\frac{\gamma(\gamma+1)}{2}M_\infty^2\left[\frac{9}{16}a_0^2\xi^{-\frac{1}{2}}+\frac{3}{2}a_0a_1\xi^{-\frac{3}{4}}+\frac{1}{4}a_1^2\xi^{-1}+\ldots\right] \quad (2.21)$$

where

$$a_0 = \frac{2}{\sqrt{3}}\left(\frac{4b}{4-3ab}\right)^{\frac{1}{4}},$$

$$a_1 = -\frac{192\,ea_0^2}{(342-297\,ab)a_0^4 - 128\,b},$$

$$a_2 = \frac{2a_0a_1[(216ab-201)a_0^2a_1 - 80e]}{(180-189ab)a_0^4 - 64b}.$$

For the case of no-slip ($e = 0$), we get,

$$\eta = a_0\xi^{\frac{3}{4}} \qquad (2.22)$$

and

$$\frac{p_w}{p_\infty} = \frac{9\gamma(\gamma+1)}{32}M_\infty^2 a_0^2 \xi^{-\frac{1}{2}}. \qquad (2.23)$$

In the following table, we compare the results of present investigation with no-slip and no suction/injection with the exact results of Stewartson [5] and the approximate results of Yasuhara [3]:

		Δ	P_w
Stewartson's results		0·704	0·555
Present investigation with linear profile		0·737	0·514
Yasuhara's results	Quadratic profile	0·738	0·602
	Quartic profile	0·822	0·756
	Sextic profile	0·892	0·890

We find that our results are slightly closer to the exact results of Stewartson [5] than those of Yasuhara [3] with quadratic profile. Thus we expect that the results will be reasonably accurate even with slip and suction/injection on the surface.

We also integrated (2.16) numerically. The numerical solution agrees with the analytical solutions in the leading edge and strong interaction regions. It also provides the solution in the transition region between leading edge and strong interaction regions.

3. DISCUSSION OF RESULTS

From Fig. 1, we find that suction decreases the boundary layer thickness while injection increases it. Also, slip velocity on the surface reduces the boundary layer thickness even when suction or injection is applied at the surface. From Fig. 2 we find that in the absence of slip, pressure tends to infinity as the leading edge is approached while slip enables the pressure to attain a finite value at the leading edge. This shows that slip is an important phenomenon to be considered near the leading edge of the flat plate. In the far off region as well, slip reduces the level of pressure. Injection increases viscous–inviscid interaction while suction reduces it.

From Fig. 3, we find that near the leading edge, slip velocity is about 90 per cent of the

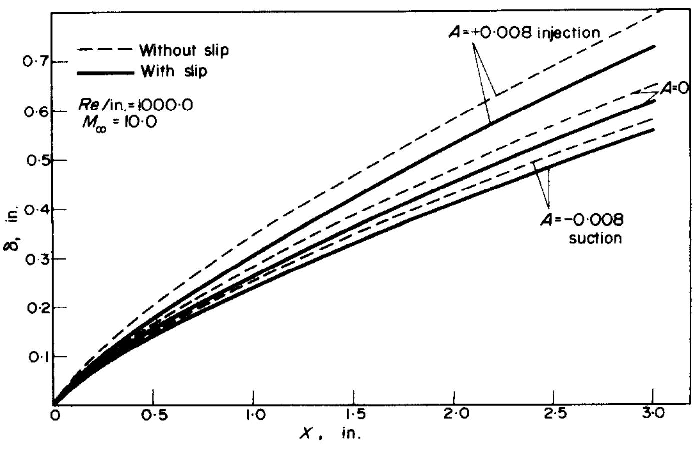

FIG. 1. Viscous layer edge versus the distance along the plate.

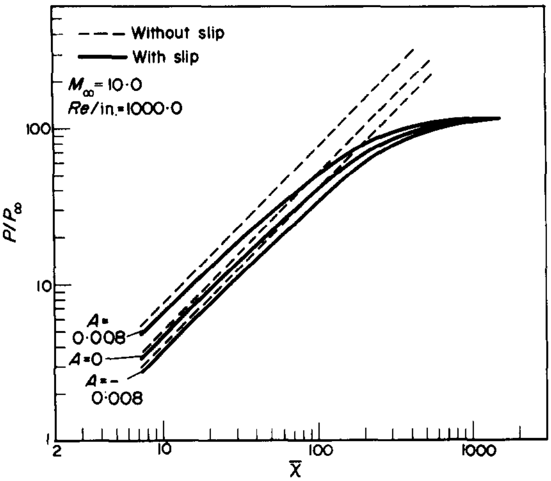

FIG. 2. Variation of p/p_∞ with $\bar{\chi}$, $\bar{\chi} = M_\infty^3(\sqrt{c})/\sqrt{Re_x}$.

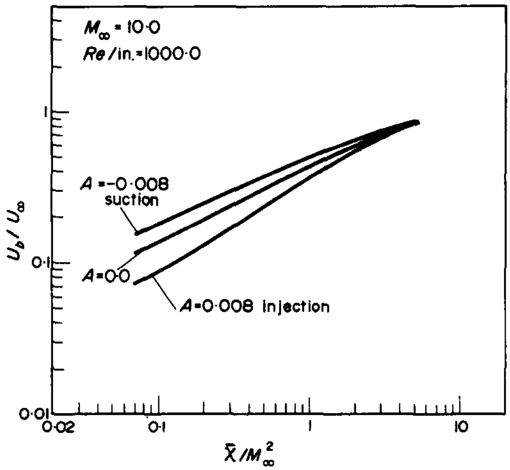

FIG. 3. Variation of slip velocity with $\bar{\chi}/M_\infty^2$ (rarefaction parameter).

free stream velocity while sufficiently far downstream, slip velocity is about 10 per cent of the free stream velocity. It is interesting to observe that suction increases the slip velocity while injection reduces it. Figure 4 shows that suction or injection velocity tends to a finite value in the leading edge region and decreases as we proceed downstream. In Fig. 5 skin friction is plotted against $(M_\infty(\sqrt{c})/\sqrt{Re_x})$. It shows that injection increases c_f in leading edge region while it decreases c_f in strong interaction region. This may be explained as follows.

Injection decreases c_f in the absence of slip. It also decreases slip velocity. Decrease in slip velocity increases c_f and vice-versa. When slip and injection both are considered, c_f changes due to injection itself and also due to change in slip velocity caused by injection. In leading edge region, since slip is dominant, increase in c_f due to decrease in slip velocity is

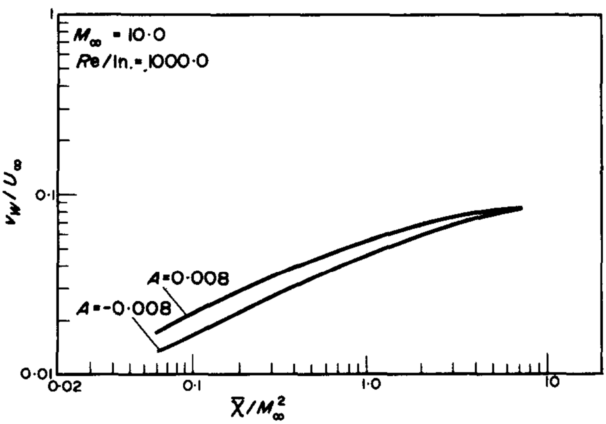

FIG. 4. Variation of suction and injection velocity with $\bar{\chi}/M_\infty^2$.

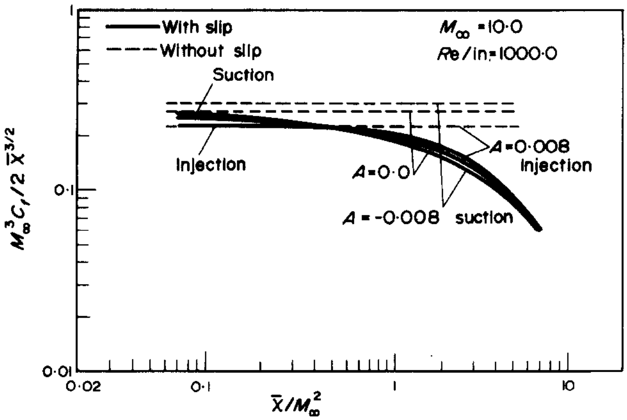

FIG. 5. Variation of $M_\infty^3 . c_f/2\bar{\chi}^{\frac{3}{2}}$ with $\bar{\chi}/M_\infty^2$.

more than decrease in c_f due to injection giving an overall increase in c_f. In strong interaction region, slip is small and so decrease in c_f due to injection dominates over increase in c_f due to decrease in slip velocity giving an overall decrease in c_f. Suction has opposite effect on c_f.

An inspection of the various graphs shows that injection has more effect on the various flow quantities than suction.

4. CONCLUSIONS

It has been shown that slip decreases boundary-layer thickness, pressure level and skin friction on the surface. Pressure attains a constant value in the leading edge region in the presence of slip while it tends to infinity in the absence of slip. Further, slip exists even in the strong interaction region.

Injection increases boundary layer thickness and pressure on the surface but decreases slip velocity. Also, injection first increases c_f in leading edge region and then decreases it in strong interaction region. Suction has opposite effect on all these flow quantities. In general, injection has larger effect on the various flow quantities than suction.

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ECOULEMENT HYPERSONIQUE RAREFIE SUR UNE PLAQUE PLANE ISOLEE AVEC SUCCION OU INJECTION

Résumé—On a utilisé la méthode de Karman–Polhausen pour étudier l'écoulement hypersonique raréfié sur une plaque plane isolée avec vitesse de glissement et succion ou injection pariétale. Des solutions analytiques sont obtenues pour le bord d'attaque et les régions à forte interaction. Les équations principales sont ensuite intégrées numériquement. Les résultats numeriques sont en accord avec les résultats analytiques au voisinage du bord d'attaque et dans les régions de forte interaction et ils donnent une solution dans la zone complémentaire. Les effets de glissement et de succion ou d'injection pour différented conditions sont discutés.

HYPERSONISCHE STRÖMUNG VERDÜNNTER GASE ÜBER EINE ISOLIERTE EBENE PLATTE MIT ABSAUGEN ODER EINBLASEN

Zusammenfassung—Zur Untersuchung der hypersonischen Strömung verdünnter Gase über eine isolierte ebene Platte mit Gleitgeschwindigkeit und Absaugen oder Einblasen an der Oberfläche wurde die Kármán–Pohlhausen-Methode angewandt. Damit erhält man analytische Lösungen für die Anströmkante und für Gebiete starker Wechselwirkungen. Die beschreibenden Gleichungen werden sodann numerisch integriert. Die numerischen stimmen mit den analytischen Ergebnissen für die Anströmkante und für Gebiete starker Wechselwirkungen überein und liefern eine Lösung für die dazwischenliegende Zone. Die Einflüsse von Gleitung und Absaugen oder Einblasen auf verschiedene Strömungseigenschaften werden erörtert.

СВЕРХЗВУКОВОЕ РАЗРЕЖЁННОЕ ОБТЕКАНИЕ ИЗОЛИРОВАННОЙ ПЛОСКОЙ ПЛАСТИНЫ ПРИ ОТСОСЕ/ВДУВЕ

Аннотация—Использовался метод Кармана-Польхаузена для исследования разреженного сверхзвукового обтекания изолированной плоской пластины при наличии скольжения и при отсосе или вдуве на поверхности. Получены аналитические решения для передней кромки и участков сильного взаимодействия. Затем полученные уравнения были проинтегрированы. Численные результаты согласуются с аналитическими как для передней кромки, так и для участков сильного взаимодействия и обеспечивают получение решения для промежуточной зоны. Рассматривается влияние скольжения, а также отсоса или вдува на различные величины потока.